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Abstract

In this paper new semiparametric GARCH models with long memory are introduced. A multiplicative decomposition of the volatility into a conditional and unconditional component is assumed. The estimation of the latter is carried out by means of a data-driven local polynomial smoother. Recurring on the revised recommendations by the Basel Committee to measure market risk in the banks' trading books, these new semiparametric GARCH models are applied to obtain rolling one-step ahead forecasts for the Value at Risk (VaR) and Expected Shortfall (ES) for market risk assets. Standard regulatory traffic light tests and a newly introduced traffic light test for the ES are carried out for all models. In addition to that, model performance is assessed via a recently introduced model selection criterion. The practical relevance of our proposal is demonstrated by a comparative study. Our results indicate that semiparametric long memory GARCH models are a meaningful substitute to their conventional, parametric counterparts.

Keywords: long memory, GARCH models, Value at Risk, Expected Shortfall, traffic light test, backtesting

JEL Classification: C14, C51, C52, G17, G32

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1 Introduction

Induced by the global financial crisis of 2007, the Basel Committee on Banking Supervision (BCBS) has published new recommendations for the measurement of market risk from the banks' trading books by means of the "Fundamental Review of the Trading Book (FRTB)" framework (Basel Committee on Banking Supervision, 2013). The review had become necessary since the global financial crisis had clearly exposed the weaknesses of the VaR (value at risk, Morgan, 1996) method, which is predominantly used by banks to measure market risk. Thus, and according to the FRTB, the Basel Committee recommends to replace the VaR method by the coherent ES (expected shortfall, Acerbi and Tasche, 2002) measure. In addition, the committee proposes to employ the proven "traffic light tests" for backtesting the model quality of the ES (Basel Committee on Banking Supervision, 1996; Basel Committee on Banking Supervision, 2016). However, backtesting is much more complex for ES than for VaR, since not only the number is relevant, but also the amount by which the expected loss is exceeded. So far there is no consensus which model or method is the most suitable for forecasting VaR and ES. Nevertheless, empirical studies have revealed that long memory GARCH (LM-GARCH) models are very successful in accurately forecasting the conditional volatility of asset returns and often outperform short memory GARCH type models (see, among others, Giot and Laurent, 2003, Degiannakis* (2004), Tang and Shieh, 2006, Grané and Veiga (2008), Härdle and Mungo (2008), Baillie and Morana, 2009, Demiralay and Ulusoy, 2014, Aloui and Ben Hamida, 2015 and Royer, 2022).

Against this background, the paper at hand focuses on the introduction and application of a new semiparametric long memory GARCH (Semi-LM-GARCH) model, which belongs to a general class of non-stationary volatility models as outlined in Sucarrat (2019). The most commonly known approaches for modelling non-constant conditional variances are the autoregressive conditional heteroscedasticity (ARCH) model proposed by Engle (1982) and its generalisation, the generalized ARCH (GARCH) model, introduced by Bollerslev (1986). Those models and their extensions imply exponentially decaying autocorrelations of the squared innovations and do not control for long memory in the conditional dynamics. However, Ding et al. (1993), Ding and Granger (1996), Andersen and Bollerslev (1997), Andersen et al. (1999) and Cotter (2005), among others, found evidence for the presence of long memory in the empirical autocorrelations in absolute or squared observations of financial time series. Analogously to the extension of the ARIMA, i.e. the fractional ARIMA (FARIMA) model introduced by Granger and Joyeux (1980), Baillie et al. (1996) proposed the fractionally integrated GARCH (FIGARCH) model which proved to be successful in modelling the long term dynamics in volatility of various financial time series (see e.g. Bollerslev and Mikkelsen, 1996, Tse, 1998, Beine et al., 2002, and Baillie and Morana, 2009. However, another branch of literature suggests that these long term dynamics might partly stem from deterministic structural shifts in the unconditional variance (see e.g. Lamoureux and Lastrapes, 1990 and Mikosch and Stărică, 2004). For instance, Beran and Ocker (2001) revealed the presence of a non-constant deterministic scale function for some volatility series by means of the semiparametric fractional autoregressive (SEMIFAR) model, introduced by Beran and Ocker (1999). Furthermore, Feng (2004) found that conditional heteroskedasticity and change in volatility usually occur simultaneously in financial return series. Under regular conditions a process with conditional heteroskedasticity is covariance stationary, but a process with change in volatility is at best locally stationary. This potentially non-stationary process can be transformed into a weakly stationary process by eliminating the deterministic component from the original process, as was illustrated by Feng (2004) and by Van Bellegem and Von Sachs (2004). The authors assume that volatility is multiplicatively decomposed into a conditional and unconditional component and that the latter changes slowly over time. They propose to estimate the time varying unconditional variance by means of a kernel smoother of the squared residuals. Engle and Rangel (2008) as well as Brownlees and Gallo (2010) apply another multiplicative decomposition based on exponential quadratic and penalised B-splines, respectively. Mazur and Pipień (2012) introduce the almost periodically correlated (APC-) GARCH. In this model the scaling function is parameterized by means of the Flexible Fourier Form by Gallant (1981, 1984). More recently, Amado and Teräsvirta (2014) introduced the Time varying GARCH model under the same assumption (see also Amado and Teräsvirta, 2008, 2013, 2017) and underline the empirical importance of considering deterministic changes in the unconditional variance of financial return series.

In analogy to Feng (2004) we introduce various semiparametric long memory GARCH (Semi-LM-GARCH) models. We propose to estimate the time varying unconditional variance by means of an adapted version of the SEMIFAR algorithm (Beran and Feng, 2002a) with a local polynomial estimator. Subsequently, the deterministic component is removed from the data and a LM-GARCH model is fitted to the approximately stationary residuals. Practical performance is first illustrated by the application to daily return series of 22 major stock indices. Moreover, our proposal is applicable to model quantitative risk measures. This is illustrated by calculating the one-step ahead out-of-sample forecasts of the VaR and ES at the 99%- and 97.5% confidence level with a forecast horizon of approximately one year, as required by the latest regulations proposed by the Basel committee (see Basel Committee on Banking Supervision, 2017). A comprehensive comparison study between conventional parametric LM-GARCH models and Semi-LM-GARCH models reveals that our proposals are an attractive alternative.

The paper is organized as follows. Section 2 recaps the most commonly known long memory GARCH models as well as a fractionally integrated version of the Log-GARCH (Geweke, 1986, Pantula, 1986, and Milhøj, 1987), the FI-Log-GARCH, which was recently proposed by Feng et al. (2020). The proposed models are introduced in Section 3. In Section 4 the semiparametric estimation of the deterministic component is illustrated, an adaptation of the SEMIFAR algorithm is briefly described and the practical implementation of our proposals with regard to rolling forecasts is discussed. Employment of our proposals to VaR and ES is explained in Section 5. Empirical results are presented in Section 6. Section 7 concludes.

2 Modelling long memory in volatility

The existence of long memory in volatility was first discovered in the S&P 500 daily closing index by Ding et al. (1993). Up to this time volatility models were assumed to have an exponentially decaying correlation of volatility. In the following decade, more researchers found evidence for the presence of long memory in volatility of financial asset prices, including intraday and high-frequency stock returns (see e.g. Ding and Granger, 1996, Andersen and Bollerslev, 1997, Andersen et al., 1999 and Cotter, 2005, among others). As a consequence, various LM-GARCH models were developed.

2.1 GARCH models with long memory

In the short memory case, the autoregressive conditional heteroscedasticity (ARCH) model proposed by Engle (1982) and its generalisation, the generalized ARCH (GARCH) model, introduced by Bollerslev (1986), are well-known approaches for modelling non-constant conditional variances. Let r_t^*, $t = 1, ..., n$, denote the (log-) returns of a stock or financial index with $E(r_t^*) = \mu_{r^*}$. A common representation of a GARCH (p, q) model is given by

$$\begin{aligned} r_t &= \sqrt{h_t}\epsilon_t, \\ h_t &= \omega + \sum_{i=1}^{p} \alpha_i r_{t-i}^2 + \sum_{j=1}^{q} \beta_j h_{t-j} \end{aligned} \tag{1}$$

where $r_t = r_t^* - \mu_{r^*}$ are the centralized returns, ϵ_t being identically independent distributed (i.i.d.) random variables with $E(\epsilon_t) = E(\epsilon_t^2 - 1) = 0$, h_t denotes the conditional variances, $\omega > 0$ and $\alpha_1, \ldots, \alpha_p, \beta_1 \ldots \beta_q \geq 0$. Due to increasing evidence for volatility series exhibiting long memory, an extension of the GARCH which captures this important feature, namely the fractionally integrated GARCH model (FIGARCH) was proposed by Baillie et al. (1996). Moreover, based on the exponential GARCH (EGARCH) proposed by Nelson (1991), Bollerslev and Mikkelsen (1996) introduced the fractionally integrated exponential GARCH (FIEGARCH), where the logarithm of the conditional variance is modelled as a fractionally integrated process. The EGARCH and FIEGARCH account for the so-called leverage effect which usually has short-term effects on the dependence structure of the underlying process. Furthermore, Ding et al. (1993) proposed the so called asymmetric power GARCH (APARCH) model. It controls for the power transformation of the volatility process and the asymmetric absolute residuals in order to avoid misspecification for non-normal data. The extension to the fractionally integrated APARCH (FIAPARCH) was then proposed by Tse (1998) which combines the FIGARCH with the APARCH. Another model that is capable of capturing persistence in volatility is the ARCH(∞) model introduced by Robinson (1991) and further investigated by Giraitis et al. (2000), Kazakevičius and Leipus (2002) as well as Douc et al. (2008). Moreover, in a recent study, Royer (2022) proposed an ARCH(∞) extension of the APARCH that accounts for conditional asymmetry in the presence of severe long memory. Its specification is very general and nests the ARCH(∞) as well as the Threshold-ARCH(∞) (see Bardet and Wintenberger, 2009).

2.2 The FI-Log-GARCH model

Recently, Feng et al. (2020) proposed the FI-Log-GARCH. This model is an extension of the independently introduced Log-GARCH model by Geweke (1986), Pantula (1986) and Milhøj (1987) and represents a symmetric special case of the FIAPARCH for $\delta \to 0$. In the following a brief derivation of the FI-Log-GARCH is given. Following Sucarrat et al. (2016) and Francq and Sucarrat (2018), the Log-GARCH is defined by (1) and

$$\ln h_t = \omega + \sum_{i=1}^{p} \alpha_i \ln r_{t-i}^2 + \sum_{j=1}^{q} \beta_j \ln h_{t-j}. \tag{2}$$

As for the EGARCH, no non-negativity constraints are needed. Define $\mu_{lr^2} = E(\ln r_t^2)$ and $\eta_t = \ln \epsilon_t^2 - \mu_{l\epsilon^2}$, where $\mu_{l\epsilon^2} = E(\ln \epsilon_1^2)$. Then following Francq and Sucarrat (2018) model (2) can be represented as an ARMA(p^*, q) with $p^* = \max(p, q)$:

$$\phi(B)(\ln r_t^2 - \mu_{lr^2}) = \psi(B)\eta_t, \tag{3}$$

where B is the backshift operator, $\phi(B) = 1 - \sum_{i=1}^{p^*} \alpha_i B^i - \sum_{j=1}^{q} \beta_j B^j$ and $\psi(B) = 1 + \sum_{j=1}^{q} \psi_j B^j = 1 - \sum_{j=1}^{q} \beta_j B^j$. Analogously to the extension of the GARCH to the FIGARCH, the extension of the Log-GARCH to the FI-Log-GARCH is straight forward (see Baillie et al., 1996 and Feng et al., 2020). Factorising the left hand side of (3) with the fractional differencing operator $(1 - B)^d$ yields

$$\phi(B)(1 - B)^d(\ln r_t^2 - \mu_{lr^2}) = \psi(B)\eta_t. \tag{4}$$

According to Hosking (1981b) and Granger and Joyeux (1980) the fractional differencing operator can be defined as

$$(1 - B)^d = \sum_{k=0}^{\infty} \theta_k(d) B^k, \tag{5}$$

where $\theta_k(d) = (-1)^k \frac{\Gamma(d+1)}{\Gamma(k+1)\Gamma(d-k+1)}$, $d \in (-0.5, 0.5)$ and $\Gamma(\cdot)$ denotes the Gamma function. In this model $\ln r_t^2$ is assumed to follow a linear FARIMA (fractional autoregressive integrated moving average) process as introduced by Hosking (1981a), and Granger and Joyeux (1980). This model is well defined, if the innovation distribution satisfies some regularity conditions, $\phi(z)$ and $\beta(z)$ have no common factors with all roots lying outside the unit circle and $0 < d < 1/2$. The Log-GARCH is a special case with $d = 0$. Let $\alpha_{d,i} = \psi(B) - \phi(B)(1 - B)^d$. Then the conditional volatility of the FI-Log-GARCH can be obtained by

$$\ln h_t = \omega + \sum_{i=1}^{\infty} \alpha_{d,i} \ln r_{t-1}^2 + \sum_{j=1}^{q} \beta_j \ln h_{t-j}, \tag{6}$$

where ω is some constant. (6) is a long memory extension of (2) with hyperbolically decaying coefficients $\alpha_{d,i}$. For a more detailed derivation and a comprehensive illustration of the properties of the FI-Log-GARCH we refer the reader to Feng et al. (2020).

3 Semiparametric extension of the FI-Log-GARCH

In this section, we introduce a semiparametric extension of the FI-Log-GARCH, i.e. the Semi-FI-Log-GARCH, which belongs to a general class of non-stationary volatility models as outlined in Sucarrat (2019). We follow Feng (2004) as well as Van Bellegem and Von Sachs (2004) and assume a multiplicative decomposition of h_t into a conditional and unconditional component under the assumption that the latter is slowly varying over time. While Feng (2004) as well as Van Bellegem and Von Sachs (2004) propose to estimate the unconditional variance by smoothing $\{r_t^2\}$ via a kernel estimator, we employ a local polynomial smoother.

3.1 The Semi-FI-Log-GARCH model

In order to control for a time varying unconditional variance we add a non-negative smooth deterministic function $\sigma(\tau_t)$ into (1) and obtain

$$r_t = \sqrt{\sigma(\tau_t) h_t} \epsilon_t, \tag{7}$$

where $\tau_t = (t - 0.5)/n$ denotes the rescaled time. Define $\xi_t^2 = r_t^2/\sigma(\tau_t) = h_t \epsilon_t^2$. $\{\xi_t\}$ is assumed to follow a FI-Log-GARCH process given by

$$\begin{aligned} \xi_t &= \sqrt{h_t} \epsilon_t \quad \text{and} \\ \ln h_t &= \alpha_0 + \sum_{i=1}^{\infty} \alpha_{d,i} \ln \xi_{t-i}^2 + \sum_{j=1}^{q} \beta_j \ln h_{t-j}. \end{aligned} \tag{8}$$

Hence, (7) together with (8) represent a Semi-FI-Log-GARCH model. To ensure that our model is well defined we assume that $\text{var}(\xi_t) = 1$ and $E(\ln \xi_t^2)$ exists which implies that

$\xi_t \neq 0$ almost surely. Furthermore, we have $g(\tau_t) = \ln \sigma(\tau_t) + \mu_{l\xi^2}$ and $Z_t = \ln \xi_t^2 - \mu_{l\xi^2} = \ln h_t + \eta_t - (\mu_{l\xi^2} - \mu_{l\epsilon^2})$, where $\mu_{l\xi^2} = E(\ln \xi_t^2)$ and $\eta_t = \ln \epsilon_t^2 - \mu_{l\epsilon^2}$. Let $Y_t = \ln r_t^2$. The log-transformation of r_t^2 admits an additive regression model with a determinsitic, non-parametric function and is of the form

$$Y_t = g(\tau_t) + Z_t, \tag{9}$$

where Z_t follows a FARIMA (p^*, q) given by

$$\phi(B)(1-B)^d Z_t = \psi(B)\eta_t. \tag{10}$$

We see that the Semi-FI-Log-GARCH is equivalent to a SEMIFAR (semiparametric fractional autoregressive) model (Beran and Feng, 2002c; Beran and Ocker, 1999) with an additional MA-part and the restriction $p \geq q$. Subsequently, well developed SEMIFAR algorithms are applicable for estimating $g(\tau_t)$ and Z_t. Moreover, we have $\hat{\sigma}(\tau_t) = \hat{C}_\sigma \exp[\hat{g}(\tau_t)]$ and $\hat{h}_t = \hat{C}_h \exp(\hat{Z}_t)$, where $C_\sigma = \exp(-\mu_{l\xi^2})$ and $C_h = \exp(\mu_{l\xi^2} - \mu_{l\epsilon^2})$ can be estimated consistently by the scale-adjusted returns and through standardizing the innovations under the assumptions that $E(\xi_1^4) < \infty$ and $\mathrm{var}\,(\xi_t) = 1$.

3.2 Other Semi-LM-GARCH models

The conditional volatility in (7) can also be modelled by means of any conventional LM-GARCH under the assumption that there are strictly stationary solutions for all these models. Hence, in the following we formulate three new semiparametric LM-GARCH models. The Semi-FIGARCH is specified by

$$h_t = \omega^* + \{1 - \psi^{-1}(B)\phi(B)(1-B)^d\}\xi_t^2, \tag{11}$$

where $\omega^* = \omega\psi^{-1}(B)$ and with all roots of $\phi(B)$ and $\psi(B)$ outside the unit circle. Moreover, the Semi-FIEGARCH is given by

$$\ln h_t = \omega + \phi^{-1}(B)(1-B)^{-d}\alpha(B)g(\epsilon_{t-1}), \tag{12}$$

where $\alpha(B) = 1 + \sum_{i=1}^{p} \alpha_i B^i$ and $g(\epsilon_t) = \Theta\epsilon_t + \gamma[(|\epsilon_t| - E|\epsilon_t|)]$ is the news impact function with Θ, $\gamma \in \mathbb{R}$. And, subsequently, the Semi-FIAPARCH is defined by

$$h_t^\delta = \omega + \{1 - \psi^{-1}(B)\phi(B)(1-B)^d\}(|\xi_t| - \gamma_i\xi_t)^\delta, \tag{13}$$

where $\delta > 0$, $\gamma_i < |1|$ with $i = 1, ..., p$. δ is a parameter to be estimated and γ_i controls for an asymmetric response of the volatility to positive and negative shocks. Please note that for the FI-Log-GARCH the long memory parameter is not affected by a power- or log-transformation (see Surgailis and Viano, 2002, and Lemma 2 in Feng et al. (2020). However, this is yet to be proven for other LM-GARCH models. Nonetheless, estimation of the deterministic component for models (11), (12) and (13) is carried out analogously to the Semi-FI-Log-GARCH under the assumptions that $\{\xi_t^2\}$ is a log-linear process and d is not affected by the log-transformation.

3.3 Related approaches

Several previous studies exist which employ a similar methodology to decompose h_t and estimate $\sigma(\tau_t)$. To begin with, Engle and Rangel (2008) as well as Brownlees and Gallo (2010) apply another multiplicative decomposition based on exponential quadratic and penalised B-splines, respectively. Mazur and Pipień (2012) propose to parametrize the deterministic component by means of the Flexible Fourier Form by Gallant (1981, 1984). Moreover, Amado and Teräsvirta (2008, 2013, 2014, 2017) introduce the Multiplicative-Time-Varying GARCH model in which the deterministic component is modelled via generalised logistic transition functions. Furthermore, Engle et al. (2013) suggest the use of mixed-data sampling (MIDAS) in order to decompose the volatility into a short-run and a long-run component (MIDAS) and propose the GARCH-MIDAS model. And, more recently, Zhang et al. (2017) introduces the Box-Cox Semi-GARCH model. The authors suggest to estimate the deterministic scale function from the Box-Cox transformed series $|r_t|^\lambda$ instead of r_t^2.

Overall, the paper at hand contributes to the literature since the majority of the aforementioned studies do not address the persistence in the stochastic component, i.e. h_t. In addition, we propose to estimate the deterministic component, i.e. $\sigma(\tau_t)$, via a local polynomial smoother within the scope of an adapted version of the SEMIFAR algorithm (see Beran and Feng, 2002a and Letmathe et al., 2021 for the original and adapted version of the algorithm, respectively).

4 Estimation and practical implementation

The SEMIFAR model introduced by Beran and Feng (2002c) is capable of simultaneously identifying a deterministic scale as well as short- and long-range dependence. The estimation processes in two parts, namely the nonparametric estimation of the deterministic component and the parametric estimation of the parameters that determine short- and long-range dependence as well as integer differencing. Furthermore, Beran et al. (2015) proposed the exponential semiparametric FARIMA (ESEMIFAR) model under the assumption that $\{\xi_t\}$ is log-linear. As it was already shown that a FI-LOG-GARCH can be formalised as a FARIMA (p^*, q) given by (4), $\{\xi_t\}$ is indeed a log-linear process. Subsequently, this parametrization allows that (4) or (10) can be estimated by means of any standard `R` package for FARIMA models and that well developed SEMIFAR algorithms are applicable for estimating the deterministic component $g(\tau_t)$ as well.

4.1 Local polynomial smoothing

In this paper a local polynomial estimator for $g^{(\nu)}(\tau_t)$, the $\nu-$th derivative, is considered (see e.g. Beran and Feng, 2002a, Beran and Feng, 2002b, Beran and Feng, 2002c, and Beran et al., 2013). Under the assumption that g is at least $(l+1)$-times differentiable at a point t_0, $g(\tau_t)$ can be approximated by a local polynomial of order l for τ_t in a neighbourhood of τ_0. The approximation is given by

$$g(\tau_t) = g(\tau_0) + g^{(1)}(\tau_0)(\tau_t - \tau_0) + ... + g^{(l)}(\tau_0)(\tau_t - \tau_0)^l/l! + R_l, \tag{14}$$

where R_l denotes a remainder term. Following Gasser and Müller (1979), we define the weight function to be a kernel of order two with compact support $[-1, 1]$ having the polynomial form $K(x) = \sum_{i=0}^{r} a_i x^{2i}$, for $(|x| \leq 1)$,where $K(x) = 0$ if $|x| > 1$, $r \in (0, 1, 2, ...)$ and a_i are such that $\int_{-1}^{1} K(x)dx = 1$ holds. $\hat{g}^{(\nu)}$ $(\nu \leq l)$ can now be obtained by solving the locally weighted least squares problem

$$Q = \sum_{i=1}^{t} \left[Y_t - \sum_{j=0}^{l} \beta_j (\tau_i - \tau_0)^j \right]^2 K\Big(\frac{\tau_i - \tau_0}{b}\Big), \tag{15}$$

where b denotes the bandwidth and $K[(\tau_i - \tau_0)/b]$ are the weights ensuring that only observations in the neighbourhood of τ_0 are used. Consider the case where $l - \nu$ is odd.

Define $m = l + 1$, then we have $m \geq \nu + 2$ and $m - \nu$ is even. A point τ is said to be in the interior for each $\tau_t \in [b, 1 - b]$, at the left boundary if $\tau_t \in [0, b)$ and at the right boundary if $\tau_t \in (1 - b, 1]$. Following Beran and Feng (2002b) a common definition for an interior point is $\tau = cb$ with $c = 1$ and for a boundary point we have $c \in [0, 1)$. Beran and Feng (2002a) and Beran and Feng (2002b) obtained asymptotic expressions for the bias, variance and mean integrated squared error (MISE) of $\hat{g}$. According to Theorem 1 in Beran and Feng (2002b) the bias and variance are given by

$$E(\hat{g}^{(\nu)} - g^{(\nu)}) = b^{m-\nu} \frac{g^{(m)}(\tau)\beta_{(\nu,m,c)}}{m!} o(b^{(m-\nu)}), \tag{16}$$

and

$$(nb)^{1-2\mu(\tau} b^{2\nu} \text{var}\,[\hat{g}^{(\nu)}] = V(c) + o(1). \tag{17}$$

For an interior point Beran and Feng, 2002b presented a simple explicit expression of $V(1)$, which is given by $V(1) = 2\pi c_f \int_{-1}^{1} K^2_{(\nu,m)(\mu(\tau)} dx$, for $d = 0$ and for $d > 0$ we have $V(1) = 2c_f \Gamma(1 - 2d) \sin(\pi d) \int_{-1}^{1} \int_{-1}^{1} K_{(\nu,m)}(x) K_{(\nu,m)}(y) |x - y|^{2d-1} dx dy$, where c_f stands for the spectral density of the ARMA part of (10) at frequency zero. In the case of antipersistence, i.e. for $d < 0$ the formula for $V(1)$ is quite complex and is omitted. Moreover, Beran and Feng (2002b) derived an explicit expression for the asymptotic MISE (AMISE) in order to determine the asymptotically optimal bandwidth which can be obtained by

$$b_{opt} = C_{opt} n^{(2d-1)/(2m+1-2d)}, \tag{18}$$

with

$$C_{opt} = \left(\frac{[m!]^2}{2(m-\nu)} \frac{(2\nu + 1 - 2d)}{\beta^2} \frac{(d_b - c_b)V(1)}{I[g^{(m)}]} \right)^{1/(2m+1-2d)}, \tag{19}$$

where $I(g^{(m)}) = \int_{c_b}^{d_b} [g^m(\tau)] d\tau$ with $0 \leq c_b < d_b \leq 1$ being small positive constants in order to control for the boundary effect and $\beta = \int_{-1}^{1} x^m K(x) dx$. Based on these results Beran and Feng (2002a) proposed two iterative plug-in algorithms. In this paper we only consider a strongly adapted version of Algorithm B which is presented in the following.

4.2 A Plug-In Algorithm for SEMIFARIMA models

Based on the iterative plug-in (IPI) algorithms for SEMIFAR models introduced by Beran and Feng (2002a), Letmathe et al. (2021) developed an IPI-procedure for SEMIFARIMA models by translating and adapting the main features of the IPI-algorithm for SEMIFAR models from the programming language S to R, in order to enhance its overall accessibility and applicability. The algorithm processes as follows:

i) In the first iteration start with an initial bandwidth h_0 set beforehand and select p and q denoting the AR- and MA-order, respectively.

ii) Estimate g from Y_t employing h_{j-1} and calculate the residuals $\tilde{Z}_t = Y_t - \hat{g}(\tau_t)$. Estimate d and V by fitting a FARIMA (with predefined AR- and MA-order in Step i) to $\tilde{Z}_t$.

iii) Set $b_j = (b_{j-1})^{\alpha}$, where α denotes an inflation factor. Estimate $g^{(m)}$ with b_j and a local polynomial of order $l^* = l + 2$. Now, we obtain

$$b_j = \left(\frac{[m!]^2}{2m} \frac{(1-2\hat{d})}{\beta^2} \frac{(d_b - c_b)\hat{V}(1)}{I[\hat{g}^{(m)}]} \right)^{1/(2m+1-2\hat{d})} \cdot n^{(2\hat{d}-1)/(2m+1-2\hat{d})}. \tag{20}$$

iv) Repeat steps ii) and iii) until convergence or a given number of iterations has been reached and set $\hat{b}_{opt} = b_j$.

After estimating g with $\hat{b}_{opt}$ the residuals $\tilde{Z}_t$ or $\hat{\xi}_t$ can be further analysed by means of the FI-Log-GARCH or any other LM-GARCH model. Please note that the results presented in Beran and Feng (2002a,b,c), Beran and Ocker (1999, 2001), and Beran et al. (2013) remain valid for the IPI for SEMIFARIMA models. For a more detailed documentation of the procedure, changes and adaptions of this IPI-algorithm we refer the reader to Feng et al. (2021) and Letmathe et al. (2021).

4.3 Rolling one-step ahead forecasts

Volatility estimation for the conventional semi-parametric LM-GARCH models is carried out by means of the statistical software *OxMetrics* and the related `G@RCH` 8.0 package. For the Semi-FI-Log-GARCH we propose to use the free software R (R Core Team, 2021). There are various possibilities for estimating a FARIMA with R. In this paper we utilise the `fracdiff()` function from the package having the same name. The volatility is then derived from the estimates of the conditional means. The latter can be obtained via a truncated AR(∞) representation of the fitted model which is given by

$$\hat{Z}_t = \sum_{i=1}^{L} \hat{\lambda}_i \tilde{Z}_{t-i}, \quad \tilde{Z}_t = 0 \quad \text{for} \quad t < i, \tag{21}$$

where L denotes the number of lags, $\hat{\lambda}_i$ are the coefficients of $\hat{\lambda}(B) = (1-B)^{\hat{d}}\hat{\phi}(B)\hat{\beta}^{-1}(B) = 1 - \sum_{i=1}^{L} \hat{\lambda}_i B^i$ with $i = 1, .., L$ and $\tilde{Z}_t = Y_t - \hat{g}(\tau_t)$. We follow the three step-estimation procedure to obtain the total volatility as proposed by Sucarrat (2019) and described in Section 3.3., with minor adjustments. To begin with, in this paper the deterministic component $g(\tau_t)$ in (9) is estimated non-parametrically via a data-driven local polynomial smoother.

i) Estimate $g(\tau_t)$ in (9) by means of the IPI for SEMIFARIMA models and calculate the residuals $\tilde{Z}_t = Y_t - \hat{g}(\tau_t)$.

ii) Fit a FARIMA(p^*, q) model as in (10) to $\tilde{Z}_t$ and compute $\hat{Z}_t$ with (21).

iii) The total volatilities are then obtained by

$$\begin{aligned}\hat{\zeta}_t &= \sqrt{\hat{C}_\sigma} \exp[\hat{g}(\tau_t)/2] \sqrt{\hat{C}_h} \exp[\hat{Z}_t/2] \\ &= \sqrt{\hat{\sigma}(\tau_t)\hat{h}_t}.\end{aligned} \tag{22}$$

Analogously, the conditional volatility can be calculated by replacing (22) with $\sqrt{\hat{h}_t} = \sqrt{\hat{C}_h} \exp(\hat{Z}_t/2)$. Note in this context the parameters C_σ and C_h are equivalently estimated as in Sucarrat (2019) (see also Sucarrat et al., 2016 and Escribano and Sucarrat, 2018). We yield $\hat{C}_\sigma = \text{var}\left[\exp(\tilde{Z}_t/2)\right] = \exp(-\hat{\mu}_{l\xi^2})$ and $\hat{C}_h = \text{var}\left[\hat{\xi}_t/\exp(\hat{Z}_t/2)\right] = \exp(\hat{\mu}_{l\xi^2} - \hat{\mu}_{l\epsilon^2})$, where $\hat{\mu}_{l\xi^2} = -\ln[\frac{1}{n}\sum_{i=1}^{n} \exp(\tilde{Z}_t)]$ and $\hat{\mu}_{l\epsilon^2} = -\ln[\frac{1}{n}\sum_{i=1}^{n} \exp(\hat{\eta}_t)]$ are smearing estimators (see Duan, 1983) of $E(\ln \xi_t^2)$ and $E(\ln \epsilon_t^2)$, respectively. Further note that calculating the total volatilities in step **iii)** only requires an estimate of $\mu_{l\epsilon^2}$ since $\hat{C}_\sigma \hat{C}_h = \exp(-\hat{\mu}_{l\epsilon^2})$. Moreover, the rolling one-day forecasts are given by

$$\hat{Z}_{n+k} = \sum_{i=1}^{L} \hat{\lambda}_i \tilde{Z}_{n+k-i}, \tag{23}$$

where $k = 1, ..., K$, with K being the number of out-of-sample observations. We propose to extrapolate the last estimate of the deterministic component $g(\tau_t)$ for the in-sample as a forecast for the unconditional standard deviations for the out-of-sample period. That means $\tilde{Z}_{n+k-i} = Y_t - \hat{g}(\tau_t)$, if $n + k - i \leq n$ and $\tilde{Z}_{n+k-i} = Y_{n+k-i} - \hat{g}(\tau_n)$, otherwise. Rolling one-day forecasts for the total volatility can then be obtained by plugging $\hat{Z}_{n+k}$ and $\hat{\sigma}(\tau_n)$ into (22).

5 Application to VaR and ES

It has been already shown that the elimination of the deterministic component from the data can improve the estimation of quantitative risk measures (see e.g. Peitz, 2016, and Feng, forthcoming). However, GARCH and Semi-GARCH models lack the possibility of an underlying long memory structure in the conditional dynamics of daily returns. Consequently, the employment of Semi-LM-GARCH models can further improve the estimation quality of quantitative risk measures. Our approach proceeds as follows. A Semi-LM-GARCH model is fitted to an in-sample return series with $n_{\text{in}} = n - K$, trading days, where $K = 250$ denotes the number of trading days of one year. Then the out-of-sample one-step ahead forecasts of the VaR and ES with a forecast horizon of K are being calculated with confidence levels of $\alpha_V = 99\%$ for VaR and $\alpha_E = 97.5\%$ for ES, as proposed by the Basel Committee (Basel Committee on Banking Supervision, 2016, Basel Committee on Banking Supervision, 2017).

5.1 One-day ahead forecasts of VaR and ES

Please note that we only consider the conditional t-distribution with degree of freedom $\nu > 2$. For simplicity we propose to use the last estimate of $\sigma(\tau_t)$ as the forecast for the unconditional standard deviations for the out-of-sample period such that $\hat{\zeta}_{n_{\text{in}}+k} = \sqrt{\hat{\sigma}(\tau_{n_{in}})\hat{h}_{n_{\text{in}}+k}}$. Then we have for the one-day rolling forecasts of VaR

$$\widehat{\text{VaR}}_{n_{\text{in}}+k}(\alpha) = -\bar{r} + \hat{\zeta}_{n_{\text{in}}+k} F_{\hat{\nu}}^{-1}(\alpha)\sqrt{(\hat{\nu}-2)/\hat{\nu}}, \tag{24}$$

where $k = 1, ..., K$ and F_ν denotes the cumulative distribution function of a t-distribution with variance $\nu/(\nu - 2)$, $\hat{\nu}$ is an estimate of the degrees of freedom ν and $\bar{r}$ is the sample mean of the in-sample returns. The one-day rolling forecasts of the ES are given by

$$\text{ES}_{n_{\text{in}}+k}(\alpha) = -\bar{r} + \hat{\zeta}_{n_{\text{in}}+k}\text{ES}_{\epsilon,\alpha}, \tag{25}$$

where $\hat{\mathrm{ES}}_\epsilon(\alpha)$ is the ES of a standardized t-distribution with unit variance. According to Eq. (2.25) in McNeil et al. (2015) we have, for $\nu > 2$,

$$\mathrm{ES}_\epsilon(\alpha) = \frac{f_\nu[F_\nu^{-1}(\alpha)]}{1-\alpha} \frac{\nu + [F_\nu^{-1}(\alpha)]^2}{\nu - 1} \sqrt{\frac{\nu-2}{\nu}}, \tag{26}$$

where f_ν is the density function of a t distribution. It can be shown that under the conditional t-distribution we have $\mathrm{ES}_\epsilon(\alpha) \mathrel{\hat{=}} \mathrm{VaR}(\alpha^*)$, where $\alpha^*(\nu) = F_\nu[\mathrm{ES}_\epsilon(\alpha)\sqrt{\nu/(\nu-2)}]$. For $\alpha = 0.975$ we have α^* being marginally larger but almost equal to 0.99. Consequently, $\widehat{\mathrm{ES}}(0.975)$ is slightly larger than $\widehat{\mathrm{VaR}}(0.99)$. For conventional models $\hat{h}_{n_{\mathrm{in}}+k}$ can be obtained by means of *OxMetrics*, whereas for the Semi-FI-Log-GARCH $\hat{h}_{n_{\mathrm{in}}+k}$ can directly be calculated with (23) where the coefficients can be obtained by means of any statistical programming language capable of estimating FARIMA models.

5.2 Backtesting VaR and ES

In this paper we propose carrying out a traffic light test for the VaR as stipulated by the Basel Committee on Banking Supervision (2016), which is based on the number of violations, i.e. where the losses exceed VaR estimates. Let

$$I_{n_{\mathrm{in}}+k} = \begin{cases} 1, & \text{if} \quad -r_{n_{\mathrm{in}}+k} > \widehat{\mathrm{VaR}}_{n_{\mathrm{in}}+k}(\alpha) \\ 0, & \text{otherwise,} \end{cases} \tag{27}$$

be an empirical hit sequence. Then the number of violations $\{I_{n_{\mathrm{in}}+k}(\alpha) = 1\}$ will be denoted by N_1 at $\alpha = 97.5\%$ and N_2 at $\alpha = 99\%$. In line with the Basel Committee on Banking Supervision (2016) the green zone for VaR at $\alpha = 97.5\%$ is set to $0 \leq N_1 \leq 10$ and for VaR at $\alpha = 99\%$ it is stipulated to $0 \leq N_2 \leq 4$. Then we have $\mu_1 = E(N_1) = 6.25$ and $\mu_2 = E(N_2) = 2.5$. Furthermore, we adapt the idea of Costanzino and Curran (2018) for backtesting ES. Define

$$\hat{\epsilon}^*_{n_{\mathrm{in}}+k} = -(r_{n_{\mathrm{in}}+k} - \bar{r})/\hat{\zeta}_{n_{\mathrm{in}}+k} * \sqrt{\nu/(\nu-2)}. \tag{28}$$

In order to satisfy the conditions required by Eq. (14) in Costanzino and Curran (2018) it is assumed that ϵ^*_{n+k} are i.i.d. t-distributed random variables. Moreover, we define

$$w_{n_{\mathrm{in}}+k} = \begin{cases} 1 - \frac{1-F_\nu(\epsilon^*_{n_{\mathrm{in}}+k})}{1-\alpha}, & \text{if} \quad I_{n_{\mathrm{in}}+k} = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{29}$$

Then the test statistic for the ES is a weighted sum of the negative returns that exceed $\widehat{\text{VaR}}_{n_{\text{in}}+k}(\alpha)$ and is given by

$$T_{\text{ES}} = \sum_{k=1}^{K} w_{n_{\text{in}}+k}. \tag{30}$$

Please note that according to Costanzino and Curran (2015), $(T_{\text{ES}} - \mu_T)/\sqrt{K}$ is asymptotically $\sim N(\mu_T, \sigma_t^2)$ normally distributed with $\mu_T = (1-\alpha)K/2$ and $\sigma_T^2 = (1-\alpha)(1+3\alpha)/12$. For $\alpha = 0.975$ and $K = 250$, Costanzino and Curran (2018) approximated the asymptotic boundary of the green zone for T_{ES} (with a cumulative probability till 95%) with 5.48. Moreover, the authors derived a finite-sample distribution for T_{ES} and obtained a finite sample boundary for the green zone of 5.70. In the following it will be illustrated that this new traffic light approach works very well in practice. Subsequently, we will solely focus on backtesting ES based on T_{ES}. A model is considered to pass our backtest if N_1, N_2 and T_{ES} are all situated in the green zone. The best model is then identified by means of a newly developed selection criterion introduced by Feng et al. (2020) which is defined by

$$\text{WAD} = |N_1 - \mu_1|/\mu_1 + |N_2 - \mu_2|/\mu_2 + |T_{\text{ES}} - \mu_T|/\mu_T. \tag{31}$$

An advantage of using the WAD (weighted absolute deviation) criterion instead of conventional loss functions (see e.g. Sarma et al., 2003) is that the WAD-score of a model usually does not contradict with its corresponding backtest result. The larger the quantities N_1, N_2 and T_{ES} are, the larger the WAD is. By contrast, the so called firm's loss function becomes smaller the larger N_1, N_2 and T_{ES} are.

6 Empirical results

We fit the Semi-FI-Log-GARCH (SFIL), Semi-FIGARCH (SFIG), Semi-FIEGARCH (SFIE), Semi-FIAPARCH (SFIA) and their parametric equivalents (FIL, FIG, FIE and FIA) with respect to 22 return series of major stock indices over the period from January 1999 to December 2019. A time span of 20 years is sufficiently large and includes all relevant financial crise (and especially the global financial market crisis from 2007) with possible long term volatility dynamics. Consequently, the implementation of long memory models particularly for forecasting VaR and ES is justified and consistent. Historical return series of major stocks indices from Asia, Europe and America are considered in order to evaluate the performance of our proposals in different world markets. We employ the following stock indices: AEX Index (AEX), ATHEX Comp. (ATH), Austrian Trades Index (ATX), CAC 40 (CAC), S&P/TSX Composite (CAD), DAX 30 (DAX), Dow Jones Industrial (DJI), EURO STOXX 50 (EST), FTSE 100 (FTS), Ireland SE (ISQ), Hang Seng (HSI), Korea SE Comp. (KOR), Madrid SE General (MAD), Mexico IPC (MEX), NASDAQ Comp. (NSQ), Nikkei 225 (NIK), NYSE (NYS), OMX Stockholm (OMX), Portugal PSI General (PSI), Russel 2000 (RUS), Standard and Poor 500 (S&P) and Swiss Market (SWI). We split the data into training (in-sample) and test-sample (out-of-sample). The training set contains $N - K$ observations, with N being the number of total observations and $K = 250$ days for the test-sample. Initially, local polynomial trend estimation by means of the IPI for SEMIFARIMA models, which is introduced in section 4.2, is applied to the training data with regard to the semiparametric models. Subsequently,the parametric parts of the models are fitted to the trend-adjusted return series in order to calculate the one-day rolling forecasts for the test-samples. The Semi-FI-Log-GARCH is fitted by using the *R* package `fracdiff` and the rolling forecasts are manually calculated according to equation (23). The parametric components of the other models and the rolling forecasts are fitted and obtained by means of the `G@RCH` 8.0 package which is implemented in *OxMetrics*. All models are fitted with order $(1, d, 1)$ and are tested at coverage probabilities 97.5% and 99% under conditional normal- and t-distribution, as required by the Basel Committee on Banking Supervision (2016).[1]

6.1 Fitted model parameters

The estimated parameters $\hat{\phi}$, $\hat{\psi}$, the long memory parameter $\hat{d}$ and the degrees of freedom $\hat{\nu}$ for all semi-parametric and parametric models are shown in Tables 3 and 4 [2]. For the SFIE and FIE, sign- and magnitude effects are additionally shown and denoted by $\hat{\theta}_1$ and $\hat{\theta}_2$. Finally, $\hat{\gamma}$ and $\hat{\delta}$ represent the leverage effect and transformation parameter of SFIA and FIA, respectively. As shown, $\hat{\phi}$ and $\hat{\psi}$ for SFIG, SFIA and SFIL are comparable in terms of their range of values. However, $\hat{d}$ of SFIG is usually larger and shows values larger than 0.5. The estimated parameters for SFIE deviate substantially. For this model,

[1] We do not present results from tests assuming a conditional normal distribution, but provide them on request.

[2] Standard errors are omitted to save space but are available upon request.

$\hat{d}$ is negative and the absolute values of $\hat{\phi}$ and $\hat{\psi}$ are clearly larger than the corresponding estimates for the other semi-parametric models in most cases. Moreover it is shown, that $\hat{d}$ exhibits smaller values in the semi-parametric models as compared to the parametric counterparts (see Table 6)[3]. This is due to the fact, that the deterministic component is removed from the original process before the stochastic part is further analysed by means of a parametric model. As a consequence, if an underlying deterministic scale function is ignored, it could be falsely captured as persistence or short-term dependence in the data. This can lead to unstable volatility predictions, which in turn translates to risk measures. A significant smaller $\hat{d}$ can be observed for all SFIL models and surprisingly, the long memory parameter is negative for all SFIE models although it is above 0.5 in most cases for all FIE models (see Tables 3, 4 and 6). The fitted FIE models deliver surprising results in some cases. In particular with regard to the AEX and HSI series, it seems that the FIE extremely underestimates (overestimates) the AR component, while these obviously biased estimates are somewhat corrected by the SFIE. In most of the cases, it is not possible to estimate SFIA or FIA models as the estimation algorithm is not converging, which might be caused by the in-sample size (although being relatively large) still being too small.[4] Nevertheless, the estimated model parameters of the SFIL, SFIG, FIL and FIG are robust throughout all stock indices.

6.2 Backtesting results

The test statistics, N_1, N_2, T_{ES} and the WAD values for all semi-parametric and parametric models are listed in Table 1 and 2, respectively. A model is considered as having passed the traffic light test if its quantities N_1, N_2 and T_{ES} are all within their green zones. Among those models, the one with a minimal WAD value is determined to be the most suitable or accurate model in terms of predicting VaR and ES. Here, N_{ES} denotes the amount of how many times the out-of-sample losses exceeded the ES. However, this statistic is only provided for further information and is not considered in the traffic light test. We have $N_{\mathrm{ES}} \leq N_2$. The majority of models pass the traffic light test. In fact, only

[3]All estimates are statistically significant at conventional confidence levels.

[4]During our research it has been found that increasing the in-sample up to 30 years or more resulted in a more stable behaviour of the FIAPARCH model. Estimation results for the FIAPARCH of major stock indices, namely S&P, DJI, NSQ, RUS, DAX, EST, NIK, and FTS, are available upon request.

in case of two return series, namely DAX and EST, there is no suitable model. Overall, our results indicate that taking into account a potential underlying deterministic scale function could be crucial for passing the traffic light test. This becomes obvious in particular for the FIG and SFIG. As illustrated in Tables 1 and 2, the FIG for DJI, CAC, HSI, KOR, NYS and OMX do not pass the traffic-light test whereas the SFIG models for the same series do. The same is true for the FIE and SFIE for DJI, NSQ, CAC, NYS and OMX as well as for the FIL and SFIL for S&P and DJI.

Figure 1 illustrates the one-step rolling forecasts of the VaR at 97.5% and 99% for the DJI series. The brown and red dashed line indicate the VaR at 97.5% and 99%, respectively. The corresponding breaches are exemplified by the coloured circles and triangles. Apparently, the parametric models underestimate VaR at 99% indicated by the fact that all three models are located in the yellow zone with $N_2 = 6$ for FIG and FIE and $N_2 = 5$ for the FIL (see Table 2). In contrast, when considering a non-parametric trend in the data, the number of breaches are reduced to $N_2 = 1$ for SFIG and SFIE and $N_2 = 3$ for the SFIL (see Figure 1 and Table 1). Unfortunately, in most cases a FIA or SFIA could not be fitted which might be due to the fact that the in-sample is too small. Moreover, it is observed that $T_{\mathrm{ES}} < 5.70$ is empirically implied if both N_1 and N_2 are located within their green zones. Nonetheless, we observed some cases, where N_2 is in its yellow zones, but $T_{\mathrm{ES}} < 5.70$ is still satisfied. However, if both quantities are clearly settled in their yellow zones, T_{ES} will also be in the same zone, which indicates that T_{ES} is a useful statistic for backtesting the ES.

Furthermore, the proposed WAD model selection criterion measures the overall forecasting quality of a fitted model. An overview of the WAD values for all models across all series is given in Table 5. As shown, the FIG has the lowest WAD for NSQ, ATX and ISQ whereas the SFIG performs best for S&P, DAX, EST, AEX, CAC, NYS and OMX . The FIE is superior for CAD, MAD as well as SWI while the SFIE could not outperform the other models. The FIA performs best for MEX as well as PSI while the SFIA performs best for HSI. The FIL exhibits minimum WAD values for NIK as well as RUS while the SFIL shows minimum WAD values for DAX, FTS, DJI, ATH and KOR. In total, 12 semi-parametric models and 10 parametric models have a minimum WAD. However, the models for DAX and EST did not pass the traffic light test (see Tables 1 and 2).

Overall, our results show that Semi-LM-GARCH models, and particularly the SFIG and SFIL, provide a convincing alternative to conventional parametric models when measuring market risk. Especially, if a parametric model does not perform well it is worthwhile to check whether a semi-parametric model might deliver better results.

7 Conclusion

The paper at hand introduces different classes of Semi-LM-GARCH models, which are able to improve the simultaneous modelling of conditional heteroskedasticity and a slowly changing unconditional variance. We employ a SEMIFARIMA model with a local polynomial smoother, in order to estimate the deterministic component. Bandwidth selection is carried out by means of the SEMIFARIMA-algorithm, which was recently translated from the programming language S to R by Letmathe et al. (2021).

We employ our Semi-LM-GARCH models for an out-of-sample forecasting of VaR and ES. The performance of our proposals is assessed via traffic light tests for both risk measures. In addition to that, a recently introduced model selection criterion, the WAD, is applied. Our results indicate that Semi-LM-GARCH approaches are a meaningful substitute of parametric LM-GARCH models. Against this background, the models and results at hand may help banking supervisors as well as banks to further improve the ES measure and the processes for backtesting it. Furthermore, the performance of each model is dependent on the market. Therefore, it is advisable for both, risk managers and regulators, to constantly monitor and benchmark a variety of models.

A comprehensive comparative study of our proposals and conventional methods, e.g. historical simulation, will be conducted in future research. Moreover, the accuracy of VaR and ES forecasting might be even further improved by extending our proposals with conditional distributions that allow for modelling skewness (see Iqbal et al., 2020). However, to the best of our knowledge closed-form expressions of asymmetric, fat tailed distributions for calculating the ES have only been partially established yet. Alternatively, one might bootstrap the empirical distribution function of the observed return series or estimate VaR and ES based on the empirical quantiles of the residuals (see El Ghourabi et al. (2016) and Francq and Zakoïan, 2015 for ideas along these lines).

Declaration of Interest

The authors report no conflicts of interest. The authors alone are responsible for the content and writing of the paper.

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References

Acerbi, Carlo and Dirk Tasche (2002). "Expected shortfall: a natural coherent alternative to value at risk". In: *Economic notes* 31 (2), 379–388. DOI: 10.1111/1468-0300.00091.

Aloui, Chaker and Hela Ben Hamida (2015). "Estimation and Performance Assessment of Value-at-Risk and Expected Shortfall Based on Long-Memory GARCH-Class Models". In: *Finance a Uver: Czech Journal of Economics & Finance* 65 (1).

Amado, Cristina and Timo Teräsvirta (2008). "Modelling conditional and unconditional heteroskedasticity with smoothly time-varying structure". In: *CREATES Research Paper* (2008-8). DOI: 10.2139/ssrn.1148141.

– (2013). "Modelling volatility by variance decomposition". In: *Journal of Econometrics* 175 (2), 142–153. DOI: 10.1016/j.jeconom.2013.03.006.

– (2014). "Modelling changes in the unconditional variance of long stock return series". In: *Journal of Empirical Finance* 25, 15–35. DOI: 10.1016/j.jempfin.2013.09.003.

– (2017). "Specification and testing of multiplicative time-varying GARCH models with applications". In: *Econometric Reviews* 36 (4), 421–446. DOI: 10.1080/07474938.2014.977064.

Andersen, Torben G. and Tim Bollerslev (1997). "Intraday periodicity and volatility persistence in financial markets". In: *Journal of empirical finance* 4 (2-3), 115–158. DOI: 10.1016/S0927-5398(97)00004-2.

Andersen, Torben G., Tim Bollerslev, and Steve Lange (1999). "Forecasting financial market volatility: Sample frequency vis-a-vis forecast horizon". In: *Journal of empirical finance* 6 (5), 457–477. DOI: 10.1016/S0927-5398(99)00013-4.

Baillie, Richard T., Tim Bollerslev, and Hans Ole Mikkelsen (1996). "Fractionally integrated generalized autoregressive conditional heteroskedasticity". In: *Journal of econometrics* 74 (1), 3–30. DOI: 10.1016/S0304-4076(95)01749-6.

Baillie, Richard T. and Claudio Morana (2009). "Modelling long memory and structural breaks in conditional variances: An adaptive FIGARCH approach". In: *Journal of Economic Dynamics and Control* 33 (8), 1577–1592. DOI: 10.1016/j.jedc.2009.02.009.

Bardet, Jean-Marc and Olivier Wintenberger (2009). "Asymptotic normality of the quasi-maximum likelihood estimator for multidimensional causal processes". In: *The Annals of Statistics* 37 (5B), 2730–2759. DOI: 10.1214/08-AOS674.

Basel Committee on Banking Supervision (1996). *Supervisory Framework For The Use of Backtesting in Conjunction With The Internal Models Approach to Market Risk Capital Requirements.*

– (2013). *Fundamental review of the trading book: A revised market risk framework.*

– (2016). *Minimum capital requirements for market risk.*

– (2017). *Finalizing post-crisis reforms.*

Beine, Michel, Sébastien Laurent, and Christelle Lecourt (2002). "Accounting for conditional leptokurtosis and closing days effects in FIGARCH models of daily exchange rates". In: *Applied Financial Economics* 12 (8), 589–600. DOI: 10.1080/09603100010014041.

Beran, Jan and Yuanhua Feng (2002a). "Iterative plug-in algorithms for SEMIFAR models—definition, convergence, and asymptotic properties". In: *Journal of Computational and Graphical Statistics* 11 (3), 690–713. DOI: 10.1198/106186002420.

– (2002b). "Local polynomial fitting with long-memory, short-memory and antipersistent errors". In: *Annals of the Institute of Statistical Mathematics* 54 (2), 291–311. DOI: 10.1023/A:1022469818068.

– (2002c). "SEMIFAR models—a semiparametric approach to modelling trends, long-range dependence and nonstationarity". In: *Computational Statistics & Data Analysis* 40 (2), 393–419. DOI: 10.1016/S0167-9473(02)00007-5.

Beran, Jan, Yuanhua Feng, and Sucharita Ghosh (2015). "Modelling long-range dependence and trends in duration series: an approach based on EFARIMA and ESEMIFAR models". In: *Statistical Papers* 56 (2), 431–451. DOI: 10.1007/s00362-014-0590-x.

Beran, Jan and Dirk Ocker (1999). "SEMIFAR forecasts, with applications to foreign exchange rates". In: *Journal of Statistical Planning and Inference* 80 (1), 137–153. DOI: 10.1016/S0378-3758(98)00247-X.

– (2001). "Volatility of stock-market indexes—an analysis based on SEMIFAR models". In: *Journal of Business & Economic Statistics* 19 (1), 103–116. DOI: 10.1198/07350010152472661.

Beran, Jan et al. (2013). *Long-Memory Processes.* Springer. DOI: 10.1007/978-3-642-35512-7.

Bollerslev, Tim (1986). "Generalized autoregressive conditional heteroskedasticity". In: *Journal of econometrics* 31 (3), 307–327. DOI: 10.1016/0304-4076(86)90063-1.

Bollerslev, Tim and Hans Ole Mikkelsen (1996). “Modeling and pricing long memory in stock market volatility”. In: *Journal of Econometrics* 73 (1), 151–184. DOI: 10.1016/0304-4076(95)01736-4.

Brownlees, Christian T. and Giampiero M. Gallo (2010). “Comparison of volatility measures: a risk management perspective”. In: *Journal of Financial Econometrics* 8 (1), 29–56. DOI: 10.1093/jjfinec/nbp009.

Costanzino, Nick and Michael Curran (2018). “A Simple Traffic Light Approach to Backtesting Expected Shortfall”. In: *Risks* 6 (1). DOI: 10.3390/risks6010002.

Costanzino, Nick and Mike Curran (2015). “Backtesting General Spectral Risk Measures with Application to Expected Shortfall”. In: *Journal of Risk Model Validation* 9 (1), 21–31. DOI: 10.21314/JRMV.2015.131.

Cotter, John (2005). “Uncovering long memory in high frequency UK futures”. In: *The European Journal of Finance* 11 (4), 325–337. DOI: 10.1080/13518470410001674314.

Degiannakis*, Stavros (2004). “Volatility forecasting: evidence from a fractional integrated asymmetric power ARCH skewed-t model”. In: *Applied Financial Economics* 14 (18), 1333–1342. DOI: 10.1080/0960310042000285794.

Demiralay, Sercan and Veysel Ulusoy (2014). “Value-at-risk predictions of precious metals with long memory volatility models”. In: *MPRA Paper No. 53229*.

Ding, Zhuanxin and Clive W.J. Granger (1996). “Modeling volatility persistence of speculative returns: A new approach”. In: *Journal of econometrics* 73 (1), 185–215. DOI: 10.1016/0304-4076(95)01737-2.

Ding, Zhuanxin, Clive W.J. Granger, and Robert F. Engle (1993). “A long memory property of stock market returns and a new model”. In: *Journal of Empirical Finance* 1 (1), 83–106. DOI: 10.1016/0927-5398(93)90006-D.

Douc, Randal, François Roueff, and Philippe Soulier (2008). “On the existence of some ARCH (∞) processes”. In: *Stochastic Processes and their Applications* 118 (5), 755–761. DOI: 10.1016/j.spa.2007.06.002.

Duan, Naihua (1983). “Smearing estimate: a nonparametric retransformation method”. In: *Journal of the American Statistical Association* 78 (383), 605–610. DOI: 10.1080/01621459.1983.10478017.

El Ghourabi, Mohamed, Christian Francq, and Fedya Telmoudi (2016). “Consistent Estimation of the Value at Risk When the Error Distribution of the Volatility Model is

Misspecified". In: *Journal of Time Series Analysis* 37 (1), 46–76. DOI: 10.1111/jtsa.12136.

Engle, Robert F. (1982). "Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation". In: *Econometrica* (4), 987–1007. DOI: 10.2307/1912773.

Engle, Robert F., Eric Ghysels, and Bumjean Sohn (2013). "Stock Market Volatility and Macroeconomic Fundamentals". In: *The Review of Economics and Statistics* 95 (3), 776–797. DOI: 10.1162/REST_a_00300.

Engle, Robert F. and Jose Gonzalo Rangel (2008). "The Spline-GARCH Model for Low-Frequency Volatility and Its Global Macroeconomic Causes". In: *The Review of Financial Studies* 21 (3), 1187–1222. DOI: 10.1093/rfs/hhn004.

Escribano, Alvaro and Genaro Sucarrat (2018). "Equation-by-equation estimation of multivariate periodic electricity price volatility". In: *Energy Economics* 74, 287–298. DOI: 10.1016/j.eneco.2018.05.017.

Feng, Yuanhua (2004). "Simultaneously modeling conditional heteroskedasticity and scale change". In: *Econometric Theory* 20 (3), 563–596. DOI: 10.1017/S0266466604203061.

Feng, Yuanhua, Jan Beran, and Sebastian Letmathe (2021). *esemifar: Smoothing Long-Memory Time Series*. R package version 1.0.1. URL: https://CRAN.R-project.org/package=esemifar.

Feng, Yuanhua et al. (2020). *Fractionally integrated Log-GARCH with application to value at risk and expected shortfall.* Tech. rep. Paderborn University, CIE Center for International Economics.

Francq, Christian and Genaro Sucarrat (2018). "An Exponential Chi-Squared QMLE for Log-GARCH Models Via the ARMA Representation*". In: *Journal of Financial Econometrics* 16 (1), 129–154. DOI: 10.1093/jjfinec/nbx032.

Francq, Christian and Jean-Michel Zakoïan (2015). "Risk-parameter estimation in volatility models". In: *Journal of Econometrics* 184 (1), 158–173. DOI: https://doi.org/10.1016/j.jeconom.2014.06.019.

Gallant, A Ronald (1981). "On the bias in flexible functional forms and an essentially unbiased form: the Fourier flexible form". In: *Journal of Econometrics* 15 (2), 211–245.

– (1984). "The Fourier flexible form". In: *American Journal of Agricultural Economics* 66 (2), 204–208.

Gasser, Theo and Hans-Georg Müller (1979). "Kernel estimation of regression functions". In: *Smoothing techniques for curve estimation*. Springer, 23–68.

Geweke, John (1986). "Comment". In: *Econometric Reviews* 5 (1), 57–61.

Giot, Pierre and Sébastien Laurent (2003). "Value-at-risk for long and short trading positions". In: *Journal of Applied Econometrics* 18 (6), 641–663. DOI: 10.1002/jae.710.

Giraitis, Liudas, Piotr Kokoszka, and Remigijus Leipus (2000). "Stationary ARCH models: dependence structure and central limit theorem". In: *Econometric Theory* 16 (1), 3–22. DOI: 10.1017/S0266466600161018.

Grané, Aurea and Helena Veiga (2008). "Accurate minimum capital risk requirements: A comparison of several approaches". In: *Journal of Banking & Finance* 32 (11), 2482–2492. DOI: 10.1016/j.jbankfin.2008.05.003.

Granger, Clive WJ and Roselyne Joyeux (1980). "An introduction to long-memory time series models and fractional differencing". In: *Journal of time series analysis* 1 (1), 15–29.

Härdle, Wolfgang K and Julius Mungo (2008). "Value-at-Risk and Expected Shortfall when there is long range dependence". In: *Discussion Papers from Humboldt University*.

Hosking, Jonathan R.M. (1981a). "Fractional Differencing". In: *Biometrika* 68 (1), 165–176. DOI: 10.2307/2335817.

Hosking, JRM (1981b). "Lagrange-multiplier tests of multivariate time-series models". In: *Journal of the Royal Statistical Society: Series B (Methodological)* 43 (2), 219–230.

Iqbal, Robina et al. (2020). "Multiday expected shortfall under generalized t distributions: evidence from global stock market". In: *Review of Quantitative Finance and Accounting*, 1–23. DOI: 10.1007/s11156-019-00860-1.

Kazakevičius, Vytautas and Remigijus Leipus (2002). "On stationarity in the ARCH (∞) model". In: *Econometric Theory* 18 (1), 1–16. DOI: 10.1017/S0266466602181011.

Lamoureux, Christopher G. and William D. Lastrapes (1990). "Persistence in Variance, Structural Change, and the GARCH Model". In: *Journal of Business & Economic Statistics* 8 (2), 225–234. DOI: 10.1080/07350015.1990.10509794.

Letmathe, Sebastian, Jan Beran, and Yuanhua Feng (2021). *An extended exponential SEMIFAR model with application in R*. Tech. rep. Discussion Paper. Paderborn University.

Mazur, Błażej and Mateusz Pipień (2012). "On the Empirical Importance of Periodicity in the Volatility of Financial Returns - Time Varying GARCH as a Second Order APC(2)

Process". In: *Central European Journal of Economic Modelling and Econometrics* 4 (2), 95–116.

McNeil, Alexander J., Rüdiger Frey, and Paul Embrechts (2015). *Quantitative Risk Management: Concepts, Techniques and Tools-revised edition*. Princeton University Press.

Mikosch, Thomas and Cătălin Stărică (2004). "Nonstationarities in Financial Time Series, the Long-Range Dependence, and the IGARCH Effects". In: *Review of Economics and Statistics* 86 (1), 378–390. DOI: 10.1162/003465304323023886.

Milhøj, Anders (1987). "A Conditional Variance Model for Daily Deviations of an Exchange Rate". In: *Journal of Business & Economic Statistics* 5 (1), 99–103. DOI: 10.1080/07350015.1987.10509564.

Morgan, J.P. (1996). *RiskMetrics^{TM}–Technical Document*.

Nelson, Daniel B. (1991). "Conditional Heteroskedasticity in Asset Returns: A New Approach". In: *Econometrica* 59 (2), 347–370. DOI: 10.2307/2938260.

Pantula, Sastry G (1986). "Modeling the persistence of conditional variances: a comment". In: *Econometric Reviews* 5, 79–97.

Peitz, Christian (2016). *Die parametrische und semiparametrische Analyse von Finanzzeitreihen: neue Methoden, Modelle und Anwendungsmöglichkeiten*. Springer-Verlag.

R Core Team (2021). *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing. Vienna, Austria. URL: https://www.R-project.org/.

Robinson, Peter M (1991). "Testing for strong serial correlation and dynamic conditional heteroskedasticity in multiple regression". In: *Journal of Econometrics* 47 (1), 67–84. DOI: 10.1016/0304-4076(91)90078-R.

Royer, Julien (2022). "Conditional asymmetry in Power ARCH (∞) models". In: *Journal of Econometrics*. DOI: 10.1016/j.jeconom.2021.10.013.

Sarma, Mandira, Susan Thomas, and Ajay Shah (2003). "Selection of Value-at-Risk models". In: *Journal of Forecasting* 22 (4), 337–358. DOI: 10.1002/for.868.

Sucarrat, Genaro (2019). "The log-GARCH model via ARMA representations". In: *Financial Mathematics, Volatility and Covariance Modelling: Volume 2*, 336.

Sucarrat, Genaro, Steffen Grønneberg, and Alvaro Escribano (2016). "Estimation and inference in univariate and multivariate log-GARCH-X models when the conditional density is unknown". In: *Computational Statistics & Data Analysis* 100, 582–594. DOI: 10.1016/j.csda.2015.12.005.

Surgailis, Donatas and Marie-Claude Viano (2002). "Long memory properties and covariance structure of the EGARCH model". In: *ESAIM: Probability and Statistics* 6, 311–329. DOI: 10.1051/ps:2002017.

Tang, Ta-Lun and Shwu-Jane Shieh (2006). "Long memory in stock index futures markets: A value-at-risk approach". In: *Physica A: Statistical Mechanics and its Applications* 366, 437–448. DOI: 10.1016/j.physa.2005.10.017.

Tse, Yiu Kuen (1998). "The conditional heteroscedasticity of the yen–dollar exchange rate". In: *Journal of Applied Econometrics* 13 (1), 49–55.

Van Bellegem, Sébastien and Rainer Von Sachs (2004). "Forecasting economic time series with unconditional time-varying variance". In: *International Journal of Forecasting* 20 (4), 611–627. DOI: 10.1016/j.ijforecast.2003.10.002.

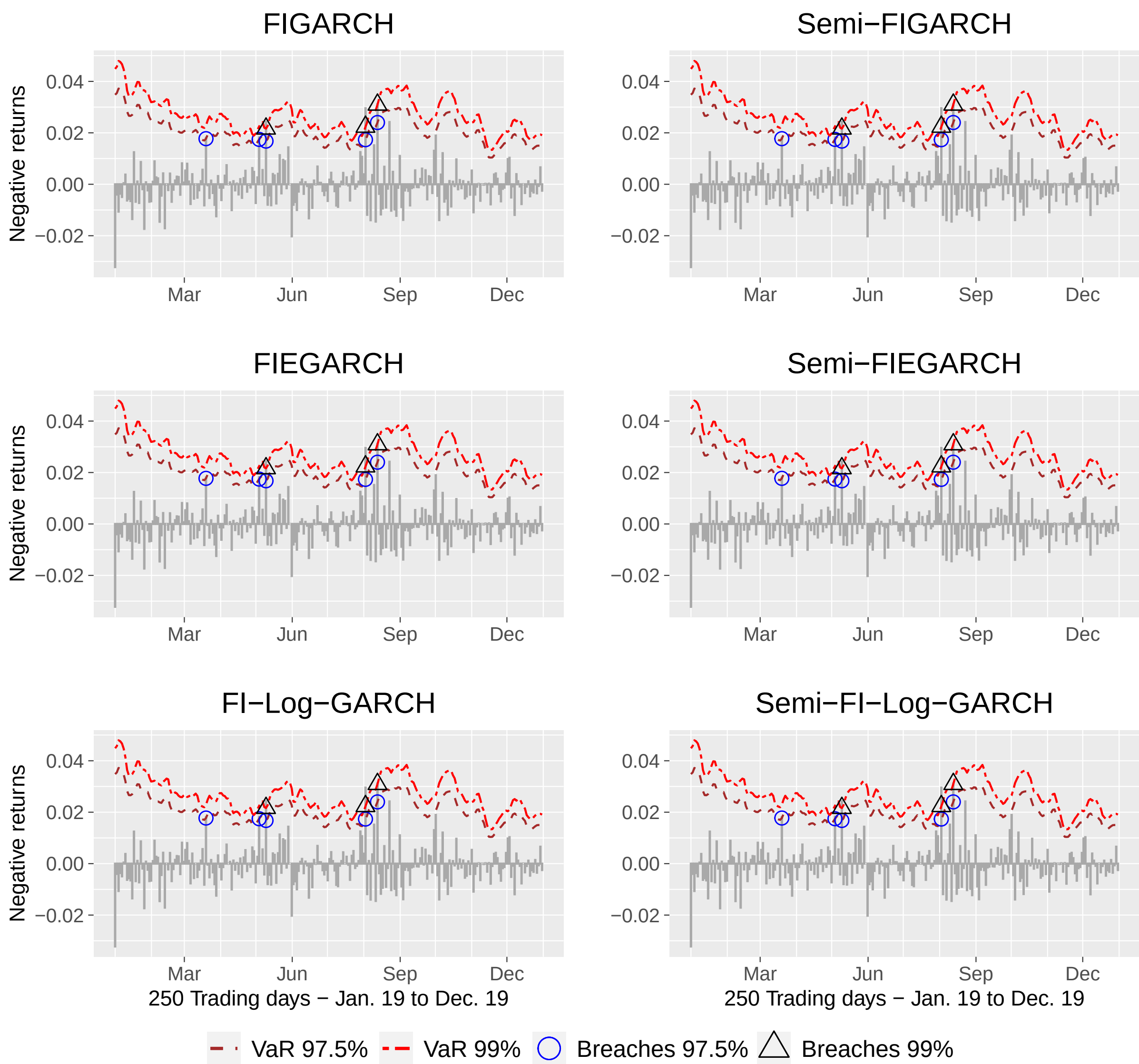

Figure 1: VaR one-step ahead forecasts at 97.5% and 99% for DJI from January 2019 to December 2019 indicated by the brown and red-dashed line, respectively.

Table 1: Results for all semi-parametric models with order (1,d,1). (Jan. 19 to Dec. 19)

Model / Series	SEMI-FIGARCH					SEMI-FIEGARCH					SEMI-FIAPARCH					SEMI-FI-LOG-GARCH				
	N_1	N_2	ES	T_{ES}	WAD	N_1	N_2	ES	T_{ES}	WAD	N_1	N_2	ES	T_{ES}	WAD	N_1	N_2	ES	T_{ES}	WAD
S&P	7	2	2	3.31	**0.38**	6	3	3	3.67	0.42	-	-	-	-	-	6	2	2	2.65	0.39
DAX	7	5	4	4.78	1.65	7	5	5	5.33	1.83	-	-	-	-	-	7	5	4	4.79	1.65
EST	11	6	6	6.37	3.20	10	7	7	6.85	3.59	-	-	-	-	-	11	7	6	6.87	3.76
NIK	6	2	2	2.35	**0.49**	3	1	1	1.41	1.67	3	1	1	1.64	1.60	3	2	2	1.38	1.28
FTS	7	5	4	4.97	1.71	7	4	4	4.58	1.18	-	-	-	-	-	6	3	3	3.51	**0.36**
RUS	5	3	3	3.15	**0.41**	4	2	2	2.11	0.89	-	-	-	-	-	3	2	2	1.76	1.16
DJI	6	1	1	2.64	0.80	5	1	1	1.54	1.31	-	-	-	-	-	5	3	2	2.44	**0.62**
NSQ	5	2	2	2.39	**0.64**	4	2	2	2.11	0.88	-	-	-	-	-	3	2	2	2.01	1.08
AEX	6	3	3	3.55	**0.37**	4	3	3	2.91	0.63	-	-	-	-	-	4	3	3	3.24	0.60
ATH	4	4	4	3.14	0.97	4	4	3	3.27	1.01	4	4	4	3.30	1.02	5	2	2	2.26	**0.68**
ATX	2	0	0	0.78	**2.43**	1	0	0	0.19	2.78	1	0	0	0.28	2.75	1	0	0	0.32	2.74
CAC	5	3	3	3.26	**0.44**	5	4	4	3.64	0.96	-	-	-	-	-	5	3	3	2.80	0.51
CAD	2	1	1	1	**1.96**	1	0	0	0.43	2.70	1	0	0	0.54	2.67	1	0	0	0.28	2.75
HSI	8	4	4	4.21	1.23	9	2	2	3.83	0.87	8	2	2	3.56	**0.62**	9	3	3	3.90	0.89
ISQ	2	0	0	0.51	**2.52**	0	0	0	0.00	3.00	0	0	0	0.00	3.00	0	0	0	0.00	3.00
KOR	8	2	2	3.76	0.68	6	2	2	2.51	0.44	-	-	-	-	-	7	2	2	3.03	**0.35**
MAD	4	3	3	2.77	0.67	4	3	3	2.47	0.77	-	-	-	-	-	5	2	1	2.38	**0.64**
MEX	2	0	0	0.22	**2.61**	1	0	0	0.35	2.73	1	0	0	0.21	2.77	1	0	0	0.12	2.80
NYS	5	1	1	2.94	**0.86**	4	2	1	2.04	0.91	-	-	-	-	-	5	1	1	1.98	1.17
OMX	8	3	3	3.48	**0.60**	8	2	2	2.33	0.73	-	-	-	-	-	4	0	0	1.26	1.96
PSI	6	2	2	2.39	**0.47**	5	2	1	2.01	0.76	5	2	2	2.00	0.76	3	1	0	1.11	1.76
SWI	5	2	2	2.61	**0.57**	5	1	1	1.67	1.27	-	-	-	-	-	2	0	0	0.45	2.54

Table 2: Results for all parametric models with order (1,d,1). (Jan. 19 to Dec. 19)

Model / Series	FIGARCH					FIEGARCH					FIAPARCH					FI-LOG-GARCH				
	N_1	N_2	ES	T_{ES}	WAD	N_1	N_2	ES	T_{ES}	WAD	N_1	N_2	ES	T_{ES}	WAD	N_1	N_2	ES	T_{ES}	WAD
S&P	7	4	3	4.53	**1.17**	7	6	6	4.79	2.05	-	-	-	-	-	6	5	4	4.17	1.37
DAX	8	6	5	5.18	2.34	9	5	5	5.41	2.17	-	-	-	-	-	7	6	5	5.15	2.17
EST	10	7	7	6.8	3.58	11	6	6	7.35	3.51	-	-	-	-	-	10	6	6	7.08	3.27
NIK	9	4	4	4.75	1.56	7	4	3	4.04	1.01	7	2	2	3.73	0.51	7	2	2	2.69	**0.46**
FTS	7	4	4	5.04	1.33	7	4	4	4.93	1.3	-	-	-	-	-	5	3	3	3.24	**0.44**
RUS	6	4	3	4.07	0.94	7	4	3	4.44	1.14	-	-	-	-	-	6	3	2	2.69	**0.38**
DJI	9	6	5	5.33	2.54	9	6	6	5.28	2.53	-	-	-	-	-	7	5	4	4.91	1.69
NSQ	6	3	3	3.74	**0.44**	6	5	5	4.49	1.48	-	-	-	-	-	4	3	3	3.14	0.56
AEX	7	3	3	3.7	**0.5**	8	3	3	4.41	0.89	-	-	-	-	-	5	3	3	3.62	0.56
ATH	4	4	4	3.41	**1.05**	4	4	3	3.41	**1.05**	4	4	4	3.46	1.07	3	1	1	1.36	1.68
ATX	4	1	1	1.77	**1.39**	1	1	1	0.73	2.21	2	1	1	0.79	2.03	2	1	1	0.84	2.01
CAC	9	5	5	5.42	2.17	9	5	5	5.54	2.21	-	-	-	-	-	8	4	3	4.59	**1.35**
CAD	4	2	2	1.79	0.99	6	2	2	2.1	**0.57**	4	2	2	1.93	0.94	1	1	0	0.63	2.24
HSI	9	5	5	5.25	2.12	8	3	3	4.23	**0.83**	8	3	3	4.95	1.07	9	3	3	5.58	1.43
ISQ	6	2	2	1.99	**0.60**	5	1	0	2.02	1.15	6	1	1	2.27	0.91	5	0	0	1.32	1.78
KOR	12	7	5	7.75	4.20	10	3	3	5.35	1.51	13	5	5	6.99	3.32	8	3	3	4.88	**1.04**
MAD	7	4	3	3.72	0.91	6	3	3	3.63	**0.40**	-	-	-	-	-	6	4	4	3.65	0.81
MEX	7	3	3	3.73	0.52	6	2	2	2.85	0.33	6	3	2	3.23	**0.27**	6	2	2	2.49	0.44
NYS	9	5	4	4.95	2.02	8	5	4	4.79	1.81	-	-	-	-	-	7	4	3	3.85	**0.95**
OMX	11	5	4	5.86	2.64	11	5	5	5.83	2.63	10	2	2	4.67	1.29	5	0	0	1.87	**1.60**
PSI	7	2	2	3.47	**0.43**	5	2	2	2.54	0.59	7	2	2	2.95	0.38	4	1	0	1.85	1.37
SWI	5	4	4	3.37	0.88	7	2	2	3.63	**0.48**	-	-	-	-	-	4	0	0	1.12	2.00

Table 3: Estimated parameters for all semi-parametric models and data examples (Jan. 99 - Dec 19)

Model / Series	SEMI-FIGARCH				SEMI-FIEGARCH						SEMI-FIAPARCH						SEMI-FI-Log-GARCH			
	$\hat{\phi}$	$\hat{\psi}$	$\hat{d}$	ν	$\hat{\phi}$	$\hat{\psi}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{d}$	ν	$\hat{\phi}$	$\hat{\psi}$	$\hat{\gamma}$	$\hat{\delta}$	$\hat{d}$	ν	$\hat{\phi}$	$\hat{\psi}$	$\hat{d}$	ν
S&P	0.039	-0.533	0.543	6.152	0.487	-0.982	-0.166	0.120	-0.222	7.631	-	-	-	-	-	-	0.137	-0.394	0.250	5.276
DAX	0.079	-0.457	0.405	8.625	0.314	-0.972	-0.141	0.122	-0.178	9.307	-	-	-	-	-	-	0.249	-0.487	0.252	7.545
EST	0.087	-0.420	0.365	8.549	0.293	-0.972	-0.162	0.105	-0.184	10.128	-	-	-	-	-	-	0.189	-0.381	0.196	7.216
NIK	-0.018	-0.267	0.325	7.550	0.874	-0.976	-0.109	0.135	-0.311	8.574	0.020	-0.236	0.744	1.054	0.294	8.238	0.151	-0.356	0.204	6.602
FTS	0.132	-0.453	0.404	8.978	-0.336	-0.839	-0.163	0.141	0.361	10.919	-	-	-	-	-	-	0.417	-0.513	0.180	5.922
RUS	0.059	-0.389	0.366	16.237	0.547	-0.978	-0.108	0.101	-0.199	19.880	-	-	-	-	-	-	0.253	-0.462	0.219	12.052
DJI	0.062	-0.573	0.567	6.393	0.464	-0.975	-0.136	0.126	-0.175	8.066	-	-	-	-	-	-	0.204	-0.467	0.278	5.789
NSQ	0.084	-0.449	0.404	8.348	-0.167	-0.844	-0.166	0.107	0.270	9.978	-	-	-	-	-	-	0.152	-0.397	0.237	6.811
AEX	0.078	-0.443	0.427	9.026	0.353	-0.982	-0.156	0.120	-0.214	11.752	-	-	-	-	-	-	0.217	-0.411	0.215	7.598
ATH	-0.029	-0.186	0.280	5.494	0.359	-0.853	-0.050	0.125	0.160	5.755	0.115	-0.383	0.359	1.319	0.341	5.861	0.187	-0.277	0.157	4.378
ATX	0.118	-0.315	0.294	8.103	0.419	-0.982	-0.107	0.167	-0.309	9.723	0.153	-0.321	0.600	1.242	0.261	9.546	0.303	-0.442	0.189	7.035
CAC	0.065	-0.382	0.353	8.857	0.175	-0.967	-0.170	0.107	-0.129	10.167	-	-	-	-	-	-	0.136	-0.353	0.214	7.564
CAD	0.168	-0.482	0.373	9.014	0.245	-0.978	-0.114	0.103	-0.169	10.974	0.291	-0.520	1.000	1.067	0.288	10.854	0.398	-0.595	0.229	7.661
HSI	0.196	-0.550	0.350	7.773	1.934	-0.970	-0.027	0.043	-0.095	8.491	0.217	-0.479	0.616	1.258	0.285	8.363	0.286	-0.535	0.227	7.216
ISQ	-0.133	-0.006	0.221	7.574	0.326	-0.952	-0.090	0.132	-0.154	8.578	0.166	-0.334	0.634	1.160	0.242	8.341	0.056	-0.181	0.138	6.978
KOR	0.034	-0.309	0.293	6.800	0.668	-0.975	-0.104	0.099	-0.254	7.363	-	-	-	-	-	-	0.219	-0.391	0.167	6.101
MAD	0.110	-0.402	0.342	8.792	0.305	-0.978	-0.132	0.120	-0.220	9.810	-	-	-	-	-	-	0.237	-0.468	0.232	7.471
MEX	0.189	-0.472	0.362	7.279	0.357	-0.986	-0.102	0.134	-0.215	8.808	0.280	-0.532	0.721	1.275	0.329	8.784	0.337	-0.557	0.254	6.586
NYS	0.049	-0.487	0.483	6.877	0.537	-0.977	-0.126	0.115	-0.186	8.578	-	-	-	-	-	-	0.128	-0.363	0.239	6.135
OMX	0.108	-0.366	0.316	10.166	-0.349	-0.871	-0.142	0.153	0.284	11.445	-	-	-	-	-	-	0.215	-0.300	0.138	7.104
PSI	0.155	-0.357	0.352	6.450	0.053	-0.974	-0.128	0.241	-0.248	7.187	0.212	-0.366	0.503	1.200	0.300	6.967	0.733	-0.741	0.117	4.821
SWI	0.145	-0.551	0.506	7.332	-0.044	-0.966	-0.173	0.157	-0.072	9.053	-	-	-	-	-	-	0.395	-0.525	0.232	5.465

Table 4: Estimated parameters for all parametric models and data examples (Jan. 99 - Dec 19)

Model / Series	FIGARCH				FIEGARCH						FIAPARCH						FI-Log-GARCH			
	$\hat{\phi}$	$\hat{\psi}$	$\hat{d}$	ν	$\hat{\phi}$	$\hat{\psi}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{d}$	ν	$\hat{\phi}$	$\hat{\psi}$	$\hat{\gamma}$	$\hat{\delta}$	$\hat{d}$	ν	$\hat{\phi}$	$\hat{\psi}$	$\hat{d}$	ν
S&P	0.054	-0.590	0.581	6.677	0.370	-0.334	-0.180	0.132	0.597	7.276	-	-	-	-	-	-	0.184	-0.586	0.406	5.085
DAX	0.074	-0.621	0.580	8.800	0.307	-0.348	-0.130	0.145	0.621	9.126	-	-	-	-	-	-	0.246	-0.642	0.421	7.208
EST	0.087	-0.599	0.549	8.506	0.238	-0.407	-0.144	0.136	0.615	9.248	-	-	-	-	-	-	0.235	-0.590	0.372	6.388
NIK	0.060	-0.488	0.467	7.665	1.064	-0.204	-0.080	0.150	0.577	7.911	0.107	-0.424	0.636	1.198	0.391	8.596	0.198	-0.500	0.310	6.548
FTS	0.128	-0.559	0.522	9.022	-0.039	-0.543	-0.156	0.143	0.591	11.072	-	-	-	-	-	-	0.397	-0.650	0.349	5.664
RUS	0.076	-0.533	0.495	15.438	0.349	-0.467	-0.104	0.112	0.590	16.725	-	-	-	-	-	-	0.266	-0.617	0.371	10.720
DJI	0.074	-0.646	0.628	6.843	0.291	-0.431	-0.137	0.150	0.602	7.445	-	-	-	-	-	-	0.210	-0.595	0.408	5.355
NSQ	0.090	-0.593	0.542	8.549	0.447	-0.232	-0.144	0.143	0.626	9.407	-	-	-	-	-	-	0.184	-0.611	0.432	6.556
AEX	0.079	-0.581	0.569	9.120	-0.963	-0.988	-0.166	0.378	0.765	7.786	-	-	-	-	-	-	0.253	-0.610	0.392	6.926
ATH	0.074	-0.425	0.421	5.619	0.444	-0.467	-0.046	0.147	0.550	5.633	0.128	-0.463	0.273	1.700	0.412	5.913	0.274	-0.482	0.282	3.867
ATX	0.156	-0.442	0.388	8.205	0.617	0.058	-0.104	0.196	0.620	9.353	0.219	-0.437	0.595	1.380	0.312	10.223	0.333	-0.647	0.378	6.614
CAC	0.094	-0.599	0.548	8.850	0.228	-0.371	-0.151	0.143	0.622	9.769	-	-	-	-	-	-	0.191	-0.554	0.370	7.044
CAD	0.155	-0.667	0.582	8.626	-0.068	-0.632	-0.086	0.136	0.612	9.489	0.255	-0.623	0.592	1.429	0.440	9.648	0.337	-0.724	0.434	6.609
HSI	0.155	-0.707	0.554	7.693	3.818	-0.469	-0.011	0.030	0.680	7.741	0.187	-0.649	0.305	1.681	0.484	8.183	0.248	-0.694	0.436	6.985
ISQ	0.136	-0.427	0.386	7.382	0.117	-0.377	-0.090	0.171	0.613	8.125	0.241	-0.509	0.556	1.411	0.348	8.174	0.254	-0.567	0.343	6.158
KOR	0.120	-0.542	0.444	7.006	0.631	-0.387	-0.064	0.119	0.646	7.025	0.143	-0.417	0.479	1.768	0.318	6.961	0.260	-0.664	0.417	5.397
MAD	0.118	-0.551	0.486	8.814	-0.330	-0.764	-0.130	0.141	0.506	9.712	-	-	-	-	-	-	0.245	-0.632	0.400	6.923
MEX	0.205	-0.604	0.486	7.240	0.603	0.014	-0.098	0.174	0.684	8.369	0.252	-0.571	0.465	1.699	0.404	8.310	0.315	-0.671	0.398	5.938
NYS	0.060	-0.576	0.561	7.037	0.462	-0.304	-0.141	0.132	0.610	7.940	-	-	-	-	-	-	0.190	-0.566	0.391	5.570
OMX	0.164	-0.613	0.518	9.993	0.176	-0.262	-0.125	0.183	0.651	10.614	-0.085	-0.978	0.697	1.292	1.155	10.845	0.334	-0.595	0.328	6.295
PSI	0.195	-0.465	0.413	7.036	-0.180	-0.534	-0.116	0.273	0.524	6.793	0.241	-0.422	0.412	1.695	0.325	7.119	0.547	-0.720	0.311	4.196
SWI	0.138	-0.607	0.573	7.449	-0.435	-0.843	-0.182	0.170	0.413	9.073	-	-	-	-	-	-	0.382	-0.613	0.341	5.126

Table 5: WAD-values for all models. Bold printed numbers are the models with a minimum WAD-value.

Series	Model							
	FIG	SFIG	FIE	SFIE	FIA	SFIA	FIL	SFIL
S&P	1.17	**0.38**	2.05	0.42	-	-	1.37	0.39
DAX	2.34	**(1.65)**	2.17	1.83	-	-	2.17	**(1.65)**
EST	3.58	**(3.20)**	3.51	3.59	-	-	3.27	3.76
NIK	1.56	0.49	1.01	1.67	0.51	1.60	**0.46**	1.28
FTS	1.33	1.71	1.30	1.18	-	-	0.44	**0.36**
RUS	0.94	0.41	1.14	0.89	-	-	**0.38**	1.16
DJI	2.54	0.80	2.53	1.31	-	-	1.69	**0.62**
NSQ	**0.44**	0.64	1.48	0.88	-	-	0.56	1.08
AEX	0.50	**0.37**	0.89	0.63	-	-	0.56	0.60
ATH	1.05	0.97	1.05	1.01	1.07	1.02	1.68	**0.68**
ATX	**1.39**	2.43	2.21	2.78	2.03	2.75	2.01	2.74
CAC	2.17	**0.44**	2.21	0.96	-	-	1.35	0.51
CAD	0.99	1.96	**0.57**	2.70	0.94	2.67	2.24	2.75
HSI	2.12	1.23	0.83	0.87	1.07	**0.62**	1.43	0.89
ISQ	**0.60**	2.52	1.15	3.00	0.91	3.00	1.78	3.00
KOR	4.20	0.68	1.51	0.44	3.32	-	1.04	**0.35**
MAD	0.91	0.67	**0.40**	0.77	-	-	0.81	0.64
MEX	0.52	2.61	0.33	2.73	**0.27**	2.77	0.44	2.80
NYS	2.02	**0.86**	1.81	0.91	-	-	0.95	1.17
OMX	2.64	**0.60**	2.63	0.73	1.29	-	1.60	1.96
PSI	0.43	0.47	0.59	0.76	**0.38**	0.76	1.37	1.76
SWI	0.88	0.57	**0.48**	1.27	-	-	2.00	2.54
min(WAD)	3	5+(2)	3	0	2	1	2	4+(1)

Table 6: Estimated long memory parameters for all models.

Series	Model							
	FIG	SFIG	FIE	SFIE	FIA	SFIA	FIL	SFIL
S&P	0.581	0.543	0.597	-0.222	-	-	0.406	0.250
DAX	0.580	0.405	0.621	-0.178	-	-	0.421	0.252
EST	0.549	0.365	0.615	-0.184	-	-	0.372	0.196
NIK	0.467	0.325	0.577	-0.311	0.391	0.294	0.310	0.204
FTS	0.522	0.404	0.591	0.361	-	-	0.349	0.180
RUS	0.495	0.366	0.590	-0.199	-	-	0.371	0.219
DWJ	0.628	0.567	0.602	-0.175	-	-	0.408	0.278
NSQ	0.542	0.404	0.626	0.270	-	-	0.432	0.237
AEX	0.569	0.427	0.765	-0.214	-	-	0.392	0.215
ATH	0.421	0.280	0.550	0.160	0.412	0.341	0.282	0.157
ATX	0.388	0.294	0.620	-0.309	0.312	0.261	0.378	0.189
CAC	0.548	0.353	0.622	-0.129	-	-	0.370	0.214
CAD	0.582	0.373	0.612	-0.169	0.440	0.288	0.434	0.229
HSI	0.554	0.350	0.680	-0.095	0.484	0.285	0.436	0.227
ISQ	0.386	0.221	0.613	-0.154	0.348	0.242	0.343	0.138
KOR	0.444	0.293	0.646	-0.254	0.318	-	0.417	0.167
MAD	0.486	0.342	0.506	-0.220	-	-	0.400	0.232
MEX	0.486	0.362	0.684	-0.215	0.404	0.329	0.398	0.254
NYA	0.561	0.483	0.610	-0.186	-	-	0.391	0.239
OMX	0.518	0.316	0.651	0.284	1.155	-	0.328	0.138
PSI	0.413	0.352	0.524	-0.248	0.325	0.300	0.311	0.117
SWI	0.573	0.506	0.413	-0.072	-	-	0.341	0.232